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A CONSISTENT DISCRETE ELEMENTS TECHNIQUE FOR THINWALLED ASSEMBLAGES

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Abstract—A consistent matrix formulation for the discrete element technique for linear and eigenvalue problems of structures assembled from thinwalled segments with open cross section is presented. Using the solutions of homogeneous differential equations governing the static problem as deformation modes the stiffness, load, stability and mass matrix are derived. The procedure is exemplified on few appropriate examples of bending coupled with torsion, stability and vibrations. Obtained results are exact for linear static problems and an extremely close upper bound for eigenvalue problems.

Through a limiting process, it is shown that the presented technique is an extension of the procedure used for solid beam structures.

NOTATION

A	cross-sectional area
a_x, a_y	coordinates of the shear center
d	width of the plate
e	eccentricity of the axial force
E	Young's (elastic) modulus
$I_{xx}, I_{yy}, I_{\Omega\Omega}$	moments of inertia
K	torsional moment of inertia
$k^2 = GK/EI_{\Omega\Omega}$	ratio between torsional and warping rigidity
L	overall length of the beam
l	length of the element
M_x, M_y, M_Ω	bending moments and bimoment
m_x, m_y, m_D, m_Ω	distributed external moments and bimoment
N_z	normal force
P, P_{cr}	axial and buckling force
p_x, p_y, p_z	distributed external forces
Q	generalized force
q	generalized displacement
T	total torsional moment
T_s, T_Ω	Saint-Venant's and warping torque
t	time variable
V_x, V_y	transverse forces
w_0	longitudinal displacement
x, y, z	coordinate frame
β_x, β_y	cross-sectional parameters
$\gamma_i(z)$	displacement (influence) mode
δ_{ij}	Kronecker's delta
ξ, η	componental displacements
ϑ	angle between reference x and local x coordinate line
$\kappa = kl$	ratio between torsional and warping rigidity
λ, Λ	eigenvalues
μ	ratio between warping and bending rigidity
ρ	mass per unit of volume
φ	angle of rotation about z axis
Ω	normalized sectorial coordinate

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ω	natural frequency of free vibrations
$[a]$	topological matrix
$[I]$	unit matrix
$[g_{ij}]$	stability matrix
$[k_{ij}]$	stiffness matrix
$[m_{ij}]$	mass matrix
$\{Q_i\}$	force vector
$\{q_i\}$	displacement vector

1. INTRODUCTION

A CONSISTENT matrix formulation for static problems, elastic stability and dynamic response of a structure assembled from thinwalled members is established. A thinwalled member, as a basic element of such an articulate structure, is considered as a spatial system composed from plates undergoing both bending and stressing in its plane.

The theory of flexure and torsion of thinwalled members, having as a distinctive feature the occurrence of normal stresses as a result of torsion (due to the warping), has been established and elaborated heretofore by numerous authors [1], [2], etc. However, as far as the structural systems assembled from such members are concerned, little has been done. Beside Refs. [3] and [4] discussing the force (moment area) method for very simple structures, only very few attempts have been made to treat the problem on the basis of modern techniques. Since the single thinwalled member is by itself statically undetermined regardless of boundary conditions, the number of redundant forces is considerably higher than for a similar structure assembled from solid beams. This undoubtedly emphasizes the need of the development of a method suitable for the application of computers.

Very recently Krahula [5] derived the stiffness matrix for a thinwalled member. One has to mention also the paper by Renton [6] considering one of the aspects of elastic stability. Both papers however, lack the generality and consistency of corresponding procedures developed for solid beams structures (see for instance Refs. [7] or [8]).

The task was, thus, to develop a consistent and general matrix formulation of thinwalled beams applicable to all linear and linearized structural problems. Following the procedure established by numerous authors for structures composed from solid beams appeared to be the best way to achieve the goal.

2. FORMULATION OF THE PROBLEM

The general theory of thinwalled members of open cross section, as developed in its final form by Vlasov [1], is essentially based on two assumptions in addition to those normally employed in the theory of thin elastic shells.

Assuming that the cross section of a thinwalled beam remains undeformed (rigid) and that the shearing deformation in the middle surface vanishes, the number of degrees of freedom for each cross section is reduced to four. Hence, a long cylindrical or prismatic shell is replaced by a design model standing in between the shell and beam theory. Thinwalled beams treated according to such a theory are distinguished from solid beams by experiencing longitudinal strains as a result of torsion (due to warping). In other words, instead of Bernoulli's hypothesis of plane sections, a more general hypothesis governing the kinematics of the member is introduced.

Belonging to a rigid plane, the displacement of a point on the middle surface of the member can be described by three componental displacements ξ, η and w_0 in the direction of the principal axes x, y and z, and the angle of rotation φ about the longitudinal axis z. The internal forces corresponding to the prescribed deformation field may be written in terms of componental displacements as

$$\begin{aligned} N_z &= EAw_0' & M_y &= -EI_{yy}\eta'' \\ M_x &= -EI_{xx}\xi'' & T_s &= GK\varphi' \\ T_\Omega &= -EI_{\Omega\Omega}\varphi''' & M_\Omega &= -EI_{\Omega\Omega}\varphi'' \end{aligned} \tag{2.1}$$

where N_z is the axial force; M_x, M_y bending moments; $T = T_s + T_\Omega$ total torque being a vector sum of the Saint Venant's and warping torque; and M_Ω bimoment (having the dimension of force multiplied by area). E and G stand for the elastic (Young's) and shear modulus; A is the cross-sectional area; I_{xx} and I_{yy} are principal moments of inertia defined by

$$I_{xx} = \int x^2 \, \mathrm{d}A \quad \text{and} \quad I_{yy} = \int y^2 \, \mathrm{d}A. \tag{2.2}$$

K is the torsional moment of inertia approximated for thinwalled members by

$$K \approx \frac{1}{3} \sum_i d_i \delta_i^3 \tag{2.3}$$

where d_i and δ_i are the breadth and thickness of the i-th plate. $I_{\Omega\Omega}$ is the sectorial moment of inertia defined by

$$I_{\Omega\Omega} = \int \Omega^2 \, \mathrm{d}A \tag{2.4}$$

where Ω is the normalized sectorial coordinate.

Let us note that the relations (2.1) imply the orthogonality of coordinates x, y and Ω, or that in other words the normal force N_z and bending moments M_x and M_y are reduced to the center of gravity, while the torque T and transverse forces V_x, and V_y are reduced on the shear center (Fig. 1). One also notes that the transverse forces V_x and V_y cannot be defined in terms of deformations as a result of imposed assumptions about the deformation.

The equations of equilibrium in terms of componental deformations for the linear static problem (see [1] or [2]) are due to the orthogonality of x, y and Ω uncoupled.

$$\begin{aligned} EAw_0'' &= -p_z \\ EI_{xx}\xi'''' &= p_x + m_x' \\ EI_{yy}\eta'''' &= p_y + m_y' \\ EI_{\Omega\Omega}\varphi'''' - GK\varphi'' &= m_D + m_\Omega' \end{aligned} \tag{2.5}$$

where p_x, p_y and p_z are distribution loads in the directions of the x, y and z axes, while m_x, m_y, m_D and m_Ω are distributed bending moments, torque and bimoment acting upon the member. Primes denote differentiation with respect to z.

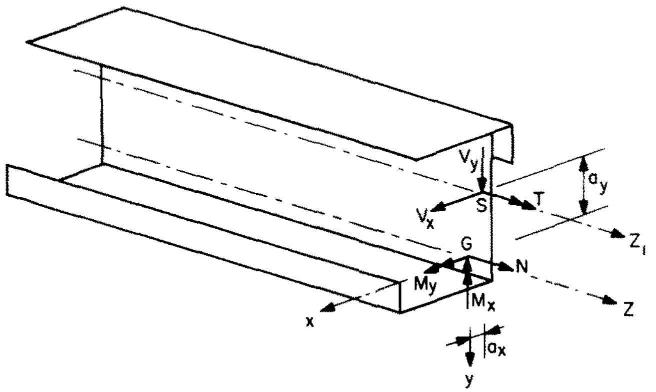


FIG. 1. A thinwalled beam where G is the center of gravity and S the shear center.

The first equation in (2.5) governs the problem of an axially stressed member while the next two govern the problem of flexure in two principal planes. The last equation (2.5d) governs the problem of warping torsion of a thinwalled member. A set of 14 boundary conditions (7 for each terminal cross section) should be attached to system (2.5) in order to define the boundary value problem.

3. DISPLACEMENT FUNCTIONS

Let us define the displacement field $q^s(z, t)$

$$q^s(z, t) = \sum_i q_i(t)\gamma_i(z) \tag{3.1}$$

in terms of some assumed displacement modes $\gamma(z)$ and unknown amplitudes $q(t)$ at structural nodes. The number of unknown nodal amplitudes $q(t)$ is chosen to equal the number of the element joint degrees of freedom. Each terminal cross section rotates about three principal axes, undergoes displacements along these axes and warps. It is only natural, then, to choose these displacements as nodal displacements q_i (Fig. 2). Moreover, we array q_i in a vector, subdivided into four subvectors

$$\{q\} = [q^z, q^x, q^y, q^\varphi]^* \tag{3.2}$$

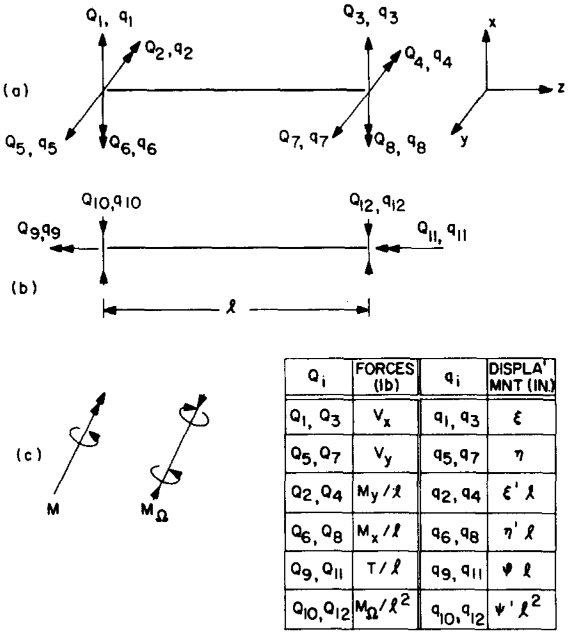
where an asterisk denotes transposition. The first two generalized displacements characterizing axial stressing of the member will in sequel be neglected in order to avoid unnecessary complications. Modes $\gamma_i(z)$ corresponding to next eight amplitudes q_i characterizing flexure of the beam are known from the literature (see for example [7] and [8]).

The solution of the last of equations (2.5) (for the homogeneous case) is

$$\varphi(z, t) = C_1 \operatorname{ch} kz + C_2 \operatorname{sh} kz + C_3 kz + C_4 \tag{3.3}$$

where

$$k_2 = GK/EI_{\Omega\Omega}. \tag{3.4}$$



Q_i	FORCES (lb)	q_i	DISPLA'MNT (IN.)
Q_1, Q_3	V_x	q_1, q_3	ξ
Q_5, Q_7	V_y	q_5, q_7	η
Q_2, Q_4	M_y/ℓ	q_2, q_4	$\xi'\ell$
Q_6, Q_8	M_x/ℓ	q_6, q_8	$\eta'\ell$
Q_9, Q_{11}	T/ℓ	q_9, q_{11}	$\psi\ell$
Q_{10}, Q_{12}	M_Ω/ℓ^2	q_{10}, q_{12}	$\psi'\ell^2$

FIG. 2. Generalized forces and displacement. (a) transverse forces and bending moments; (b) torsion moments and bimoments; (c) moment and bimoment.

In order to attain consistency the displacement nodes $\gamma_i(z)$ will be defined so that each amplitude q_i represents the total displacement at a single point. In other words γ_i is the solution to the homogeneous equation (2.5d) for $q_i = 1$ and all other $q_j = 0$ $(i \neq j)$. This leads to

$$\gamma_9(z) = \frac{1}{lD}[(1-\mathrm{ch}\,\kappa)\mathrm{ch}\,kz+\mathrm{sh}\,\kappa\,\mathrm{sh}\,kz-kz\,\mathrm{sh}\,\kappa+1-\mathrm{ch}\,\kappa+\kappa\,\mathrm{sh}\,\kappa]$$

$$\begin{aligned}\gamma_{10}(z) &= \frac{1}{\kappa l^2 D}[(\kappa\,\mathrm{ch}\,\kappa-\mathrm{sh}\,\kappa)\mathrm{ch}\,kz+(\mathrm{ch}\,\kappa-1-\kappa\,\mathrm{sh}\,\kappa)\mathrm{sh}\,kz \\ &\quad +kz(\mathrm{ch}\,\kappa-1)+\mathrm{sh}\,\kappa-\kappa\,\mathrm{ch}\,\kappa]\end{aligned} \tag{3.5}$$

$$\gamma_{11}(z) = \frac{1}{lD}[(\mathrm{ch}\,\kappa-1)\mathrm{ch}\,kz-\mathrm{sh}\,\kappa\,\mathrm{sh}\,kz+\mathrm{kz}\,\mathrm{sh}\,\kappa+1-\mathrm{ch}\,\kappa]$$

$$\gamma_{12}(z) = \frac{1}{\kappa l^2 D}[(\mathrm{sh}\,\kappa-\kappa)\mathrm{ch}\,kz+(1-\mathrm{ch}\,\kappa)\mathrm{sh}\,kz+kz+kz(\mathrm{ch}\,\kappa-1)+\kappa-\mathrm{sh}\,\kappa]$$

where

$$D = 2(1-\mathrm{ch}\,\kappa)+\kappa\,\mathrm{sh}\,\kappa \tag{3.6}$$

and

$$\kappa = kl$$

with l standing for the length of the member.

Recalling the basic knowledge of the theory of structures and in particular Mueller–Breslau theorem one defines also γ_9, γ_{11} and/or γ_{10}, γ_{12} as influence functions for reactive torsional moment and/or bimoment at both ends of a fully clamped (in sense of vanishing deplanation and twist) member.

The corresponding generalized nodal forces are chosen such that the potential energy may be written simply as a scalar product

$$V = \{q_i\}^*\{Q_i\}.$$

For the general nodal displacements as shown in Fig. 2, one has

$$\{q_i\} = [\xi_A, \xi'_A l, \xi_B, \xi'_B l, \eta_A, \eta'_A l, \eta_B, \eta'_B l, \varphi_A l, \varphi'_B l^2, \varphi_B l, \varphi'_B l^2]^*.$$

The corresponding generalized forces are

$$\{Q_i\} = [V_{xA}, M_{yA} l^{-1}, V_{xB}, M_{yB} l^{-1}, V_{yA}, M_{xA} l^{-1}, V_{yB}, M_{xB} l^{-1}, T_A l^{-1}, M_{\Omega A} l^{-2}, T_B l^{-1}, M_{\Omega B} l^{-2}].$$

Some displacements and forces are multiplied by the length of the beam l to various powers in order to achieve dimensional homogeneity. The amplitudes q_i have the dimension of length, while the generalized forces Q_i have the dimension of force.

4. THE LOAD MATRIX

The force vector Q_i has to be determined at a number of discrete points, so called structural coordinates. The load applied to the structure can, generally speaking, be either distributed or concentrated at certain locations (not necessarily coinciding with structural coordinates) or both. It is, therefore, necessary to establish a relation between the force vector $\{Q_i\}$ and external loads which will be consistent with structural idealization. In other words, the force vector Q_i should be determined to be statically equivalent to the external loading. While rule-of-the-thumb technique is to a certain extent justified for pure bending problems, one can hardly expect to apply it without significant loss of accuracy for the problems treated herein.

Having recognized the coordinate functions $\gamma_i(z)$ as influence functions, one immediately writes the relation

$$Q_i = \int_0^l \gamma_i(z) m_D(z)\,\mathrm{d}z - \int_0^l \gamma'_i(z) m_\Omega(z)\,\mathrm{d}z. \qquad (4.1)$$

In case of concentrated forces the integrals in (4.1) may be interpreted in Stieltjes' sense, or one can simply use Dirac delta functions. The same relation has been obtained by Archer [7] using virtual work argument and Betti's principle.

After some elementary calculations, one gets according to the relation (4.1) and for loads presented in Fig. 3, the load matrix given in Appendix.

5. MATRIX FORMULATION OF THE PROBLEM

The general nonlinear dynamic problem of elastic thinwalled structures presents considerable mathematical difficulties. However, a host of problems of practical significance may be treated in a much simpler form assuming the steady state motion of the structure and constancy of parametric loads. This linearization essentially transforms the nonlinear problem into an eigenvalue problem of much simpler but still complicated structure.

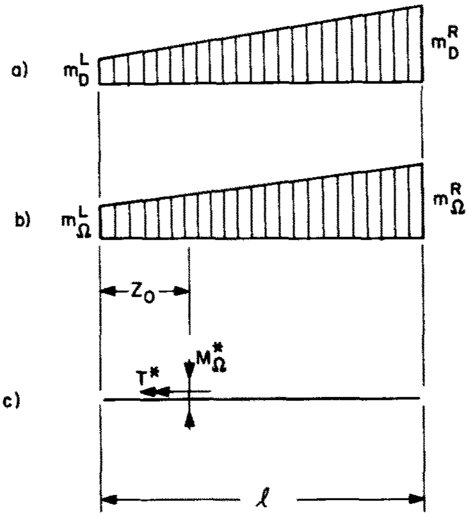


FIG. 3. External load on the member. (a) distributed torsional moment; (b) distributed bimoment; (c) concentrated torsional moment and bimoment.

In accord with the adopted model we further assume that the loads are reduced to nodes. Although this assumption is not central to the derivation, it results in a much simpler form for all the governing matrices. The linearized differential equations governing the defined eigenvalue problem [1] are

$$\begin{aligned} &EI_{xx}\xi'''' + P\xi'' + (M_x + a_y P)\varphi'' + \rho A(\ddot{\xi} + a_y\ddot{\varphi}) = 0 \\ &EI_{yy}\eta'''' + P\eta'' + (M_y - a_x P)\varphi'' + \rho A(\ddot{\eta} - a_x\ddot{\varphi}) = 0 \\ &EI_{\Omega\Omega}\varphi'''' + (r^2 P + 2\beta_x M_y - 2\beta_y M_x - GK)\varphi'' + (M_x + a_y P)\xi'' \\ &\qquad + (M_y - a_x P)\eta'' + \rho A(a_y\ddot{\xi} - a_x\ddot{\eta} + r^2\ddot{\varphi}) = 0 \end{aligned} \tag{5.1}$$

where terms in addition to those in equations (2.5) are either due to the distortion of the beam or to the inertia. External load is expressed through resultant axial force P, and resultant bending couples M_x and M_y. The coordinates of the shear center are a_x and a_y, and ρ is the product of density and gravity acceleration (per unit of volume). Other parameters are

$$\begin{aligned} r^2 &= \frac{I_{xx} + I_{yy}}{A} + a_x^2 + a_y^2 \\ \beta_x &= \frac{1}{2I_{xx}}\int x(x^2 + y^2)\,\mathrm{d}A - a_x \\ \beta_y &= \frac{1}{2I_{yy}}\int y(x^2 + y^2)\,\mathrm{d}A - a_y \end{aligned} \tag{5.2}$$

where differentiation with respect to time is indicated by dots.

We now note that the left hand sides of equation (5.1) represent the resultant forces in direction of coordinate axes x and y and the resulting torque. Multiplying them by a set of virtual displacements $\delta\xi$, $\delta\eta$ and $\delta\varphi$ and integrating over the whole domain, one gets from (5.1)

$$\begin{aligned}\int_0^l \{&[EI_{xx}\xi'''' + P\xi'' + (M_x + a_yP)\varphi'' + \rho A(\ddot{\xi} + a_y\ddot{\varphi})]\delta\xi \\ &+ [EI_{yy}\eta'''' + P\eta'' + (M_y - a_xP)\varphi'' + \rho A(\ddot{\eta} - a_x\ddot{\varphi})]\delta\eta \\ &+ [EI_{\Omega\Omega}\varphi'''' + (r^2P + 2\beta_xM_y - 2\beta_yM_x - GK)\varphi'' + (M_x + a_yP)\xi'' \\ &+ (M_y - a_xP)\eta'' + \rho A(a_y\ddot{\xi} - a_x\ddot{\eta} + r^2\ddot{\varphi})]\delta\varphi\}\,dz = 0.\end{aligned} \tag{5.3}$$

Equation (5.3) simply states that the work of all internal and external forces of an equilibrated structure vanishes for any arbitrary system of kinematically admissible virtual displacements. One also recognizes that equation (5.3) may be interpreted as Galerkin's method.

Integration by parts of equation (5.3), in conjunction with the interchange of derivation and variation, leads to

$$\begin{aligned}&\int_0^l [EI_{xx}\xi''\delta(\xi'') + EI_{yy}\eta''\delta(\eta'') + EI_{\Omega\Omega}\varphi''\delta(\varphi'') + GK\varphi'\delta(\varphi')]\,dz \\ &+ \int_0^l \{P.[\xi'\delta(\xi') + \eta'\delta(\eta')] + 2(M_x + a_yP)\varphi'\delta(\xi') + 2(M_y - a_xP)\varphi'\delta(\eta') \\ &+ (r^2P + 2\beta_xM_y - 2\beta_yM_x)\varphi'\delta(\varphi')\}\,dz \\ &+ \rho\int_0^l \{A[\ddot{\xi}\delta(\xi) + \ddot{\eta}\delta(\eta) + r^2\ddot{\varphi}\delta(\varphi)] + I_{xx}\ddot{\xi}'\delta(\xi') + I_{yy}\ddot{\eta}'\delta(\eta') \\ &+ I_{\Omega\Omega}\ddot{\varphi}'\delta(\varphi') + 2A[a_y\ddot{\varphi}\delta(\xi) - a_x\ddot{\varphi}\delta(\eta)]\}\,dz = J_0(0, l)\end{aligned} \tag{5.4}$$

where $J_0(0, l)$, usually called conjunct or concomitant, is the sum of all integrated terms. For natural boundary conditions, to be treated in sequel, conjunct vanishes and the problem is said to be self-adjoint.

Introducing for distortions $q^s(z, t)$ relation (3.1) the equation (5.4) may be rewritten in the form

$$\{\delta q_i\}([k_{ij}] - [\tilde{g}_{ij}])\{q_j\} + \{\delta q_i\}[m_{ij}]\{\ddot{q}_j\} = 0 \tag{5.5}$$

where $[k_{ij}]$, $[g_{ij}]$ and $[m_{ij}]$ are so called stiffness, stability and mass matrix. Making final use of I. Castigliano's theorem one has

$$([k_{ij}] - [g_{ij}])\{q_j\} + [m_{ij}]\{\ddot{q}_j\} = \{Q_i\} \tag{5.6}$$

where Q_i is the vector of generalized forces as defined previously.

We may note also that the most frequently used way of derivation starting from stresses and strains is not straight-forwardly applicable due to the basic assumption about the deformation.

6. STIFFNESS, STABILITY AND MASS MATRICES

The stiffness matrix $[k_{ij}]$ defined by the first integral on the left hand side of equation (5.4) is apparently a quasidiagonal matrix of 12-th order, of form

$$[k_{ij}] = \begin{bmatrix} k^{xx} & 0 & 0 \\ 0 & k^{yy} & 0 \\ 0 & 0 & k^{\varphi\varphi} \end{bmatrix}. \tag{6.1}$$

The first two submatrices representing bending stiffnesses in two principal planes, are well known from literature (see Refs. [8], [9] etc.). We will only elaborate on the third submatrix of which the elements are

$$k_{ij}^{\varphi\varphi} = EI_{\Omega\Omega} \int_0^l (\gamma_i''\gamma_j'' + k^2\gamma_i'\gamma_j')\,\mathrm{d}z \qquad (i, j = 9, \ldots, 12). \tag{6.2}$$

After two integrations by parts (6.2) reads

$$k_{ij}^{\varphi\varphi} = EI_{\Omega\Omega}[\gamma_i''\gamma_j' + \gamma_i(k^2\gamma_j' - \gamma_j''')]\big|_0^l$$

such that the final elemental submatrix $[k^{\varphi\varphi}]$ reads

$$[k^{\varphi\varphi}] = \frac{EI_{\Omega\Omega}}{Dl^5} \begin{bmatrix} \kappa^3\,\mathrm{sh}\,\kappa & \kappa^2(1-\mathrm{ch}\,\kappa) & -\kappa^3\,\mathrm{sh}\,\kappa & \kappa^2(1-\mathrm{ch}\,\kappa) \\ & \kappa(\kappa\,\mathrm{ch}\,\kappa - \mathrm{sh}\,\kappa) & \kappa^2(\mathrm{ch}\,\kappa - 1) & \kappa(\mathrm{sh}\,\kappa - \kappa) \\ & & \kappa^3\,\mathrm{sh}\,\kappa & \kappa^2(\mathrm{ch}\,\kappa - 1) \\ \text{symm.} & & & \kappa(\kappa\,\mathrm{ch}\,\kappa - \mathrm{sh}\,\kappa) \end{bmatrix} \tag{6.3}$$

where again

$$D \doteq 2(1 - \mathrm{ch}\,\kappa) + \kappa\,\mathrm{sh}\,\kappa \tag{6.4}$$

and

$$\kappa = kl\,. \tag{6.5}$$

One notes that the sign of two elements in the derived matrix is not the same as in [5]. This will be of importance in later analysis (Section 8).

The elemental stability matrix $[g_{ij}]$, as defined by the second integral in (5.4), is of more complicated form

$$[g_{ij}] = \begin{bmatrix} g^{xx} & 0 & g^{x\varphi} \\ 0 & g^{yy} & g^{y\varphi} \\ g^{\varphi x} & g^{\varphi y} & g^{\varphi\varphi} \end{bmatrix}. \tag{6.6}$$

While submatrices on the main diagonal are evidently symmetric, $[g^{\varphi x}]$ and $[g^{\varphi y}]$ are transposes of $[g^{x\varphi}]$ and $[g^{y\varphi}]$. We will restrict our attention only to matrices not presented in Ref. [8]. From (5.4) one directly reads

$$\begin{aligned} g_{ij}^{\varphi\varphi} &= \int (r^2P + 2\beta_xM_y - 2\beta_yM_x)\gamma_i'\gamma_j'\,\mathrm{d}z \qquad (i, j = 9, \ldots, 12) \\ g_{ij}^{y\varphi} &= -2\int (a_xP - M_y)\gamma_i'\gamma_j'\,\mathrm{d}z \qquad (i = 5, \ldots, 8, j = 9, \ldots, 12) \\ g_{ij}^{x\varphi} &= 2\int (a_yP + M_x)\gamma_i'\gamma_j'\,\mathrm{d}z \qquad (i = 1, \ldots, 4, j = 9, \ldots, 12) \end{aligned} \tag{6.7}$$

such that

$$
\begin{aligned}
g_{1,9} &= -g_{3,9} = -g_{1,11} = g_{3,11} = -\frac{12}{l^2\kappa^2}\left(1-\frac{\kappa^3}{12}\frac{1}{\kappa-2\,\mathrm{th}\,\frac{1}{2}\kappa}\right) \\
g_{2,9} &= g_{4,9} = -g_{2,11} = -g_{4,11} = g_{1,10} = -g_{3,10} \\
&= g_{1,12} = -g_{3,12} = \frac{6}{l^2\kappa^2}\left(1+\frac{\kappa^2}{12}-\frac{\kappa^3}{12}\frac{1}{\kappa-2\,\mathrm{th}\,\frac{1}{2}\kappa}\right) \\
g_{2,10} &= g_{4,12} = \frac{4}{l^2\kappa^2}\left[-1+\frac{\kappa}{4D}(\kappa\,\mathrm{ch}\,\kappa-\mathrm{sh}\,\kappa)\right] \\
g_{4,10} &= g_{2,12} = \frac{2}{l^2\kappa^2}\left[-1-\frac{\kappa}{2D}(\kappa-\mathrm{sh}\,\kappa)\right]
\end{aligned}
\tag{6.8}
$$

and

$$
\begin{aligned}
g_{9,9} &= -g_{9,11} = g_{11,11} = \frac{\kappa}{D^2l^3}[(1-\mathrm{ch}\,\kappa)(3\,\mathrm{sh}\,\kappa-\kappa)+\kappa\,\mathrm{sh}^2\,\kappa] \\
g_{9,10} &= -g_{10,11} = \frac{1}{D^2l^3}\left[\left(4+\frac{\kappa^2}{2}+\frac{\kappa}{2}\mathrm{sh}\,\kappa\right)(1-\mathrm{ch}\,\kappa)+2\,\mathrm{sh}^2\,\kappa\right] \\
g_{10,10} &= g_{12,12} = \frac{1}{\kappa l^3D^2}\Big[(\mathrm{ch}\,\kappa-1)(\mathrm{sh}\,\kappa+\kappa)+\kappa\,\mathrm{sh}\,\kappa \\
&\qquad\times(\kappa-2\,\mathrm{sh}\,\kappa)-\frac{\kappa^2}{2}(\kappa-\mathrm{sh}\,\kappa\,\mathrm{ch}\,\kappa)\Big] \\
g_{10,12} &= \frac{1}{\kappa l^3D^2}\left[(\mathrm{sh}\,\kappa-3\kappa)(1-\mathrm{ch}\,\kappa)+\frac{\kappa^2}{2}(\kappa\,\mathrm{ch}\,\kappa-3\,\mathrm{sh}\,\kappa)\right] \\
g_{11,12} &= -g_{9,12} = -g_{9,10}
\end{aligned}
\tag{6.9}
$$

where all coefficients (6.8) should be multiplied by $2(a_yP+M_x)$ and coefficients (6.9) by $(r^2+2\beta_xM_y-2\beta_yM_x)$. Coefficients of the submatrix $[g^{y\varphi}]$ may be easily deduced from (6.8) by increasing the first index by four and multiplying them by $-2(a_xP-M_y)$.

The mass matrix $[m_{ij}]$ defined by the third integral in equation (5.4) is of following form

$$
[m_{ij}] = \begin{bmatrix} m^{xx} & 0 & m^{x\varphi} \\ 0 & m^{yy} & m^{y\varphi} \\ m^{\varphi x} & m^{\varphi y} & m^{\varphi\varphi} \end{bmatrix}
\tag{6.10}
$$

where the first two matrices on the main diagonal are known from [9]. The elements of other submatrices are

$$
\begin{aligned}
m_{ij}^{x\varphi} &= 2\rho Aa_y\int \gamma_i\gamma_j\,\mathrm{d}z \qquad (i=1,\ldots,4, j=9,\ldots,12) \\
m_{ij}^{y\varphi} &= -2\rho Aa_x\int \gamma_i\gamma_j\,\mathrm{d}z \qquad (i=5,\ldots,8, j=9,\ldots,12) \\
m_{ij}^{\varphi\varphi} &= \rho Ar^2\int \gamma_i\gamma_j\,\mathrm{d}z \qquad (i,j=9,\ldots,12)
\end{aligned}
\tag{6.11}
$$

Such that

$$m_{1,9} = m_{3,11} = \frac{1}{lD\kappa^4}\left[\left(12\kappa - \kappa^3 + \frac{7}{20}\kappa^5\right) \text{sh } \kappa + \left(24 + \frac{\kappa^4}{2}\right)(1 - \text{ch } \kappa)\right]$$

$$m_{2,9} = -m_{2,11} = \frac{-1}{lD\kappa^4}\left[\left(6\kappa + \frac{\kappa^5}{20}\right) \text{sh } \kappa + \left(12 + \kappa^2 + \frac{\kappa^4}{12}\right)(1 - \text{ch } \kappa)\right]$$

$$m_{3,9} = m_{1,11} = \frac{1}{lD\kappa^4}\left[\left(-12\kappa + \kappa^3 + \frac{3}{20}\kappa^5\right) \text{sh } \kappa + \left(-24 + \frac{\kappa^4}{2}\right)(1 - \text{ch } \kappa)\right]$$

$$m_{4,9} = -m_{4,11} = \frac{1}{lD\kappa^4}\left[\left(-6\kappa + \frac{\kappa^5}{30}\right) \text{sh } \kappa + \left(-12 - \kappa^2 + \frac{\kappa^4}{12}\right)(1 - \text{ch } \kappa)\right] \tag{6.12}$$

$$m_{1,10} = -m_{3,12} = \frac{1}{lD\kappa^4}\left[\left(-6\kappa + \frac{3}{2}\kappa^3\right) \text{sh } \kappa + \left(12 - \kappa^2 - \frac{7}{20}\kappa^4\right) \text{ch } \kappa - 12 + \kappa^2 - \frac{3}{20}\kappa^4\right]$$

$$m_{2,10} = m_{4,12} = \frac{1}{lD\kappa^4}\left[\left(5\kappa - \frac{\kappa^3}{12}\right) \text{sh } \kappa - \left(8 + \kappa^2 - \frac{\kappa^4}{20}\right) \text{ch } \kappa + 8 + \frac{\kappa^4}{30}\right]$$

$$m_{3,10} = -m_{1,12} = \frac{1}{lD\kappa^4}\left[\left(6\kappa + \frac{\kappa^3}{2}\right) \text{sh } \kappa - \left(12 + \kappa^2 + \frac{3\kappa^4}{20}\right) \text{ch } \kappa + 12 + \kappa^2 - \frac{7}{20}\kappa^4\right]$$

$$m_{4,10} = m_{2,12} = \frac{1}{lD\kappa^4}\left[\left(\kappa + \frac{\kappa^3}{12}\right) \text{sh } \kappa - \left(4 + \frac{\kappa^4}{30}\right) \text{ch } \kappa + 4 + \kappa^2 - \frac{\kappa^4}{20}\right]$$

and

$$m_{9,9} = m_{11,11} = \frac{1}{l^2D^2\kappa}\left[(-5 \text{ sh } \kappa + 2\kappa - \kappa \text{ ch } \kappa + \kappa^2 \text{ sh } \kappa)(1 - \text{ch } \kappa) + \left(\frac{\kappa^3}{3} - 2\kappa\right) \text{sh}^2 \kappa\right]$$

$$m_{9,10} = \frac{1}{l^2D^2\kappa^2}\left[\left(\frac{7}{2}\kappa \text{ sh } \kappa - \frac{\kappa^2}{2}\right)(1 - \text{ch } \kappa) + 2\kappa^2 \text{ sh}^2 \kappa - \frac{\kappa^3}{6} \text{sh } \kappa(1 + 2 \text{ ch } \kappa)\right]$$

$$m_{9,11} = 0{\cdot}5 - m_{9,9}$$

$$m_{9,12} = \frac{1}{l^2D^2\kappa^2}\left[-\left(8 + \frac{5\kappa}{2} \text{sh } \kappa + \frac{\kappa^2}{2}\right)(1 - \text{ch } \kappa) - 4 \text{ sh}^2 \kappa + \frac{\kappa^3}{6} \text{sh } \kappa(2 + \text{ch } \kappa) - \kappa^2 \text{ sh}^2 \kappa\right]$$

$$m_{10,10} = m_{12,12} = \frac{1}{l^2D^2\kappa^3}\left[(3\kappa + 3 \text{ sh } \kappa)(1 - \text{ch } \kappa) + \frac{\kappa^3}{6}(7 + 2 \text{ ch } \kappa) - \kappa^2 \text{ sh } \kappa\left(2 + \frac{5}{2}\text{ch } \kappa\right) \quad \left(6\kappa + \frac{\kappa^3}{6}\text{sh}^2 \kappa\right)\right] \tag{6.13}$$

$$m_{10,11} = \frac{1}{l^2D\kappa^2}\left[\left(2 - \frac{\kappa^2}{2}\right)(1 - \text{ch } \kappa) + \kappa(2 \text{ sh } \kappa - \kappa \text{ ch } \kappa)\right] - m_{9,10}$$

$$m_{10,12} = \frac{1}{l^2 D^2 \kappa^3}\left[\left(5\kappa+\frac{2}{3}\kappa^3-3\,\text{sh}\,\kappa\right)(1-\text{ch}\,\kappa)-\frac{\kappa^3}{2}\text{ch}\,\kappa+\kappa^2\,\text{sh}\,\kappa \times\left(\frac{7}{2}+\text{ch}\,\kappa\right)-\kappa^3-\left(2\kappa+\frac{\kappa^3}{6}\right)\text{sh}^2\,\kappa\right]$$

$$m_{11,12} = \frac{1}{l^2 D\kappa^2}\left[-\left(2+\frac{\kappa^2}{2}\right)(1-\text{ch}\,\kappa)+\kappa(\kappa-2\,\text{sh}\,\kappa)\right]-m_{9,12}.$$

The coefficients (6.12) should be multiplied by $2\rho A a_y$, and (6.13) by $\rho A r^2$. In order to obtain elements of the submatrix $[m^{y\varphi}]$ one again increases the first index of coefficients (6.12) by four and multiplies them by $-2\rho A a_x$.

Note again that the submatrices along the main diagonal are symmetric, while $[m^{\varphi x}]$ and $[m^{\varphi y}]$ are transposes of $[m^{x\varphi}]$ and $[m^{y\varphi}]$. In final account all three matrices (stiffness, stability and mass) are symmetric reflecting the fact that the problem is self-adjoint.

Needless to say all derived matrices are computed in local coordinate system. In order to form the global matrix one employs topological matrices in the same way as for solid beam assemblages.

7. SOLUTION OF THE PROBLEM

Equation (5.5) is general enough to enable the discussion of a host of distinct problems. We will restrict ourselves to problems of static buckling (defining instability as a loss of uniqueness of solution) and free steady state vibrations. This restriction is arbitrary and stems primarily from practical considerations on the volume of the paper.

7.1 *Static buckling*

Assuming that all external (conservative) forces are characterized by a single parameter, say λ, one writes for the homogeneous case from (5.5)

$$([k_{ij}]-\lambda[g_{ij}])\{q_i\} = 0. \tag{7.1}$$

Equation (7.1) is a well known eigenvalue problem where the nontrivial solution for $\{q_i\}$ is obtained if λ is an eigenvalue of the matrix $[g_{ij}]^{-1}[k_{ij}]$. The buckling load calculated from the smallest eigenvalue $\lambda_{\min}$ should be an upper bound to the exact solution as a consequence of the applied variational principle.

7.2 *Vibration analysis*

The general vibration problem (neglecting damping) of small oscillations about the equilibrium position is governed by (5.5). For the steady state solution to the problem we seek the solution in form

$$q_i(t) = q_i \exp(i\omega t) \tag{7.2}$$

where ω is the natural frequency of the system, and i the imaginary unit. Hence, equation (5.5) may be written for the homogeneous case as

$$(-\omega^2[m_{ij}]+[k_{ij}]-[\tilde{g}_{ij}])\{q_j\} = 0. \tag{7.3}$$

The natural frequencies are now the eigenvalues of the matrix $[m_{ij}]^{-1}([k_{ij}]-[\tilde{g}_{ij}])$. For the same reason as in the previous case the obtained natural frequencies are an upper bound to the exact solution.

8. TWO LIMITING CASES

Although the analysis of an assemblage of thinwalled members according to the presented procedure does not involve special computational difficulties, it is possible to simplify it further in certain cases depending on the member geometry. The differential equation (2.5d) governing the problem of the torsion differs only by its second term on the left hand side from equations (2.5b, c) governing the flexure of the beam. The magnitude of this term is, furthermore, dependent on the parameter k (or in nondimensional form κ), being the ratio of torsional and warping rigidity. It is of interest, therefore, to analyze the influence of this term on relations derived heretofore. The analysis of two limiting cases ($\kappa \to 0$ and $\kappa \to \infty$) will not only give simple formulas, but will provide the possibility of a qualitative analysis of the problem. As it will be apparent shortly a very simple relation between the torsion and bending can be established.

8.1 *Case* $kl \to 0$

The first case to be analyzed is the case when torsional rigidity can be neglected in comparison with the warping rigidity ($GK \ll EI_{\Omega\Omega}$, i.e. $kl \approx 0$). Essentially this implies that the Saint-Venant's torsion is neglected (since only warping torsion depends on $EI_{\Omega\Omega}$). The differential equation (2.5d) has now a very simple form (same as for bending)

$$\frac{\partial^4 \varphi}{\partial z^4} = \frac{1}{EI_{\Omega\Omega}}(m_D + m'_\Omega). \tag{8.1}$$

The solution of the homogeneous equation (8.1) is obviously a cubic polynomial. Moreover, the influence functions γ_i^φ have to be identical with corresponding modes for bending γ_i^x. Using an adequate number of terms of series expansions for hyperbolic trigonometric functions sh and ch, the influence function $\gamma_9(z)$ may be written as

$$\gamma_9 = D_1/D$$

where

$$D_1 = \frac{\kappa^4}{12} - \frac{\kappa^2 (kz)^2}{4} - \frac{\kappa (kz)^3}{6} + \ldots$$

$$D = 2\left(1 - 1 - \frac{\kappa^2}{2} - \frac{\kappa^4}{24} - \ldots\right) + \kappa^2 + \frac{\kappa^4}{6} + \ldots = \frac{\kappa^4}{12} + \ldots.$$

Thus,

$$\lim_{\kappa \to 0} \gamma_9 = 1 - 3z^2/l^2 + 2z^3/l^3 \equiv \gamma_1.$$

In exactly the same way it may be shown that

$$\lim \gamma_{10} = \gamma_2; \qquad \lim \gamma_{11} = \gamma_3 \quad \text{and} \quad \lim \gamma_{12} = \gamma_4$$

when κ approaches zero.

As a result of this, for $\kappa \approx 0$, the torsional moment can be evaluated as a transverse force and bimoment as a bending moment if the external torsional load is considered as a transverse load and the bimoment load as distributed couples.

It is also easy to prove, in similar manner, that when κ approaches zero

$$\lim_{\kappa \to 0} [k^{\varphi\varphi}] = \frac{I_{\Omega\Omega}}{l^2 I_{xx}} [k^{xx}].$$

Employing the same limiting process one also finds that apart from multipliers

$$\lim_{\kappa \to 0} [\tilde{g}^{x\varphi}] = \lim_{\kappa \to 0} [\tilde{g}^{\varphi\varphi}] = [\tilde{g}^{xx}]$$

and

$$\lim_{\kappa \to 0} [\tilde{m}^{x\varphi}] = \lim_{\kappa \to 0} [\tilde{m}^{\varphi\varphi}] = [\tilde{m}^{xx}]$$

where the curl denotes that the premultiplying scalar is not considered. This means that all matrices differ only by a multiplicator having the same form as for solid beams.

Let us point out that this parallelism was responsible for our choice of displacement modes, which eventually led to the different signs in some of the stiffness coefficients when compared with [5].

It should be noted that this limiting case, besides being a useful and simple approximation (when applicable), justifies entirely the presented procedure as being nothing but an extension of established procedures for the computation of solid frames.

8.2 *Case* $\kappa \to \infty$

The second case to be briefly analyzed is the case when the warping rigidity is negligible in comparison with the torsional rigidity GK. For large values of κ one has

$$\operatorname{sh} \kappa \approx \operatorname{ch} \kappa \approx \tfrac{1}{2} e^{\kappa}$$

and

$$e^{\kappa} \gg \kappa \gg e^{-\kappa}$$

For this case the first stiffness coefficient in the submatrix $[k^{\varphi\varphi}]$ is

$$k_{9,9} = \kappa^3 \operatorname{sh} \kappa / D = \frac{\frac{1}{2}\kappa^3 \exp(\kappa)}{2(1 - \frac{1}{2}\exp\kappa) + \frac{1}{2}\kappa \exp(\kappa)} \approx \kappa^3/(\kappa - 2).$$

Finally the stiffness matrix is

$$[k^{\varphi\varphi}] = \frac{GK}{l^3 \kappa(\kappa - 2)} \begin{bmatrix} \kappa^2 & -\kappa & -\kappa^2 & -\kappa \\ & \kappa - 1 & \kappa & 1 \\ & & \kappa^2 & \kappa \\ \text{symm.} & & & \kappa - 1 \end{bmatrix}.$$

The elements of stability and mass matrix are also easily evaluated using the same argument.

9. EXAMPLES

9.1 *Bending of a plane grid*

As a first example, we consider a simple plane grid assembled from two thinwalled segments as shown in Fig. 4. The elements of vectors Q_i and q_i are arrayed somewhat

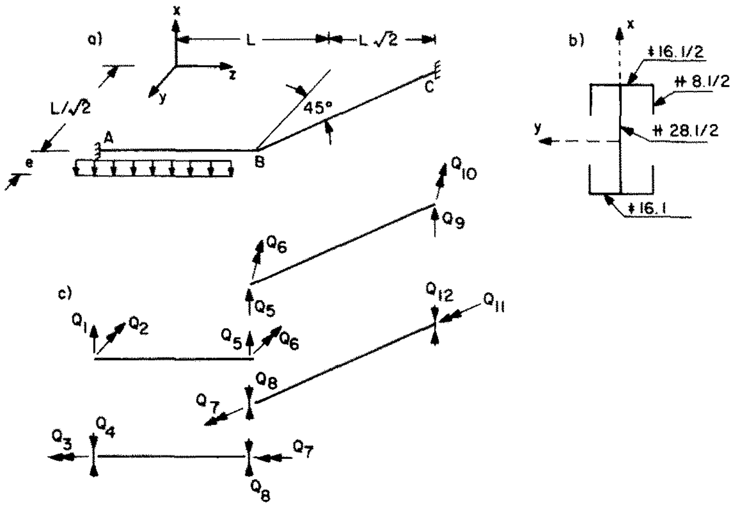


FIG. 4. (a) plane thinwalled grid subjected to bending and torsion; (b) cross section of members; (c) forces in local coordinate systems.

differently (Fig. 4c). The parameters of the cross section (shown in Fig. 4b) are

$$I_{xx} = 4540 \text{ in.}^4, \qquad I_{\Omega\Omega} = 565{,}789 \text{ in.}^6, \qquad K = 13{\cdot}2 \text{ in.}^4.$$

Therefore

$$k^2 = GK/EI_{\Omega\Omega} = 8{\cdot}9504\ 10^{-6} \text{ in.}^{-2}$$

and for $l = 120$ in., one has $\kappa = kl = 0{\cdot}36$.

Also

$$\mu = I_{\Omega\Omega}/I_{xx}l^2 = 0{\cdot}5029\ 10^{-2}.$$

The stiffness matrix for the element AB is thus

$$[k] = \frac{EI_{xx}}{l^3}\begin{bmatrix} 12 & -6 & & & -12 & -6 & & \\ -6 & 4 & & & 6 & 2 & & \\ & & 0{\cdot}060830 & -0{\cdot}030089 & & & -0{\cdot}060830 & -0{\cdot}030089 \\ & & -0{\cdot}030089 & 0{\cdot}020098 & & & 0{\cdot}030089 & 0{\cdot}009991 \\ -12 & 6 & & & 12 & 6 & & \\ -6 & 2 & & & 6 & 4 & & \\ & & -0{\cdot}060830 & 0{\cdot}030089 & & & 0{\cdot}060830 & 0{\cdot}030089 \\ & & -0{\cdot}030089 & 0{\cdot}009991 & & & 0{\cdot}030089 & 0{\cdot}020098 \end{bmatrix}.$$

In order to obtain the stiffness matrix for the element BC in global coordinates, a topological matrix $[a]$ should be used ($\vartheta = 45°$)

$$[a] = \begin{bmatrix} 1 & & & & & & & \\ & \cos\vartheta & \sin\vartheta & & & & & \\ & -\sin\vartheta & \cos\vartheta & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & \cos\vartheta & \sin\vartheta & \\ & & & & & -\sin\vartheta & \cos\vartheta & \\ & & & & & & & 1 \end{bmatrix}$$

such that

$$[k_{ij}] = [a]^*[k_{ij}][a].$$

Since the grid is rigidly clamped at A and C, such that $q_1 = q_2 = q_3 = q_4 = 0$ and $q_9 = q_{10} = q_{11} = q_{12} = 0$, one writes only the central part of the global stiffness matrix (obtained by the superposition of corresponding parts of elemental stiffness matrices written in global coordinates)

$$[k_{ij}] = \frac{EI_{xx}}{l^3}\begin{bmatrix} 24\cdot0 & 1\cdot75734 & -4\cdot24266 & 0 \\ & 6\cdot03041 & 1\cdot96859 & 0\cdot02128 \\ & & 2\cdot09124 & 0\cdot00881 \\ \text{symm.} & & & 0\cdot04020 \end{bmatrix}.$$

The distributed load p_x can be replaced through equivalent nodal loads Q_i, via the load matrix

$$\{Q\} = \begin{bmatrix} V_a \\ M_{ya}l^{-1} \\ T_a l^{-1} \\ M_{\Omega a}l^{-2} \\ V_b \\ M_{yb}l^{-1} \\ T_b l^{-1} \\ M_{\Omega b}l^{-2} \end{bmatrix} = \begin{bmatrix} 0\cdot5 & \\ -0\cdot0833 & \\ & 0\cdot5 \\ & -0\cdot0832 \\ 0\cdot5 & \\ 0\cdot0833 & \\ & 0\cdot5 \\ & 0\cdot0832 \end{bmatrix} \begin{bmatrix} -pl \\ -pe_y \end{bmatrix} = lp \begin{bmatrix} -0\cdot5 \\ 0\cdot08333 \\ -0\cdot05 \\ 0\cdot00832 \\ -0\cdot5 \\ -0\cdot08333 \\ -0\cdot05 \\ -0\cdot00832 \end{bmatrix}$$

since $e_y = 0\cdot1\ l$.

The displacements at end B are

$$\{q_b\} = [k_{ij}]^{-1}\{Q_b\} = \frac{pl^4}{EI_{xx}}\begin{bmatrix} -0\cdot07388 \\ 0\cdot09373 \\ -0\cdot26123 \\ -0\cdot19930 \end{bmatrix}.$$

Finally, the forces in local coordinates are solved from $\{q_b\}$. The diagrams of bending moments, torques and bimoments are plotted in Fig. 5.

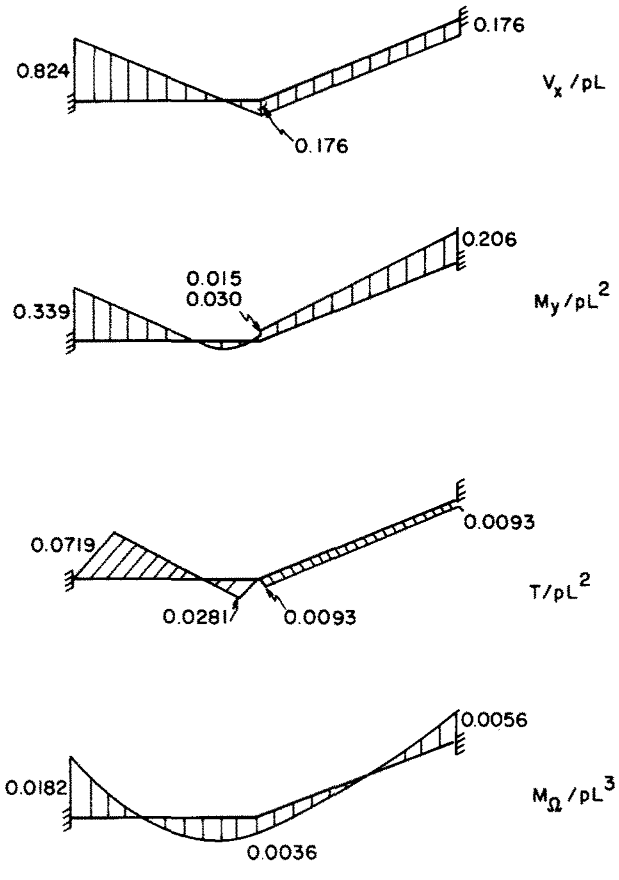


FIG. 5. Final diagrams of transverse forces, bending moments, torsional moments and bimoments.

9.2 *Elastic stability of a simply supported beam*

In order to make the comparison of results of the exact theory (Ref. [1]) and the results of the proposed discrete elements technique possible, consider a simply supported beam (Fig. 6), with a cross section given in Fig. 6b.

The cross-sectional parameters are

$$A = 26 \text{ in.}^2$$

$$I_{xx} = 165{\cdot}3 \text{ in.}^4 \qquad I_{yy} = 1702{\cdot}5 \text{ in.}^4$$

$$a_y = 4{\cdot}34 \text{ in.} \qquad a_x = 0$$

$$I_{\Omega\Omega} = 7432{\cdot}16 \text{ in.}^6 \qquad K = 5{\cdot}17 \text{ in.}^4$$

$$L = 2l = 18 \text{ ft.} = 216 \text{ in.}$$

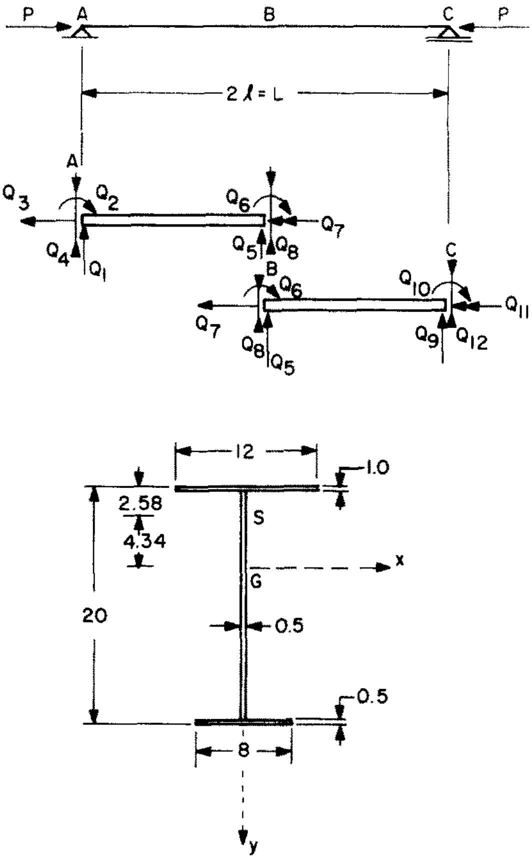


FIG. 6. Simply supported beam acted upon by an axial force P at the center of gravity G. Shear center is labelled by S.

Therefore,

$$K^2 = GK/EI_{\Omega\Omega} = 2{\cdot}674 \,.\, 10^{-4} \text{ in.}^{-2}$$

$$\kappa = kl = 1{\cdot}767$$

$$\mu = I_{\Omega\Omega}/I_{xx}l^2 = 0{\cdot}38547 \,.\, 10^{-2}.$$

The submatrices $[g^{x\varphi}]$ and $[g^{\varphi\varphi}]$ as calculated from equations (6.8) and (6.9) are

$$[g^{x\varphi}] = 2\frac{a_y P}{l^2}\begin{bmatrix} 1{\cdot}19573 & -0{\cdot}09787 & -1{\cdot}19573 & -0{\cdot}09787 \\ & 0{\cdot}12842 & 0{\cdot}09787 & -0{\cdot}03031 \\ & & 1{\cdot}19573 & 0{\cdot}09787 \\ \text{symm.} & & & 0{\cdot}12842 \end{bmatrix}$$

$$[g^{\varphi\varphi}] = \frac{r^2 p}{l^3}\begin{bmatrix} 1{\cdot}19110 & -0{\cdot}09623 & -1{\cdot}19110 & -0{\cdot}09623 \\ & 0{\cdot}12372 & 0{\cdot}09623 & -0{\cdot}02748 \\ & & 1{\cdot}19110 & 0{\cdot}09623 \\ \text{symm.} & & & 0{\cdot}1272 \end{bmatrix}$$

The submatrix $[k^{\varphi\varphi}]$ reads

$$[k_{\varphi\varphi}] = \frac{EI_{\Omega\Omega}}{l^5}\begin{bmatrix} 15.73486 & -6.30629 & -15.73486 & -6.30629 \\ & 4.40095 & 6.30629 & 1.90534 \\ & & 15.73486 & 6.30629 \\ \text{symm.} & & & 4.40095 \end{bmatrix}.$$

Since

$$r^2 = I_{xx}+I_{yy}/A+a_x^2+a_y^2 = 91.37\ \text{in.}^2$$

one has

$$2a_y/l = 0.08037$$

$$r^2/l^2 = (91.37/108)^2 = 0.72140.$$

The stiffness and stability matrices for the element AB are thus,

$$[k_{AB}] = \frac{EI_{xx}}{l^3}\begin{bmatrix} 12 & -6 & & & -12 & -6 & & \\ & 4 & & & 6 & 2 & & \\ & & 0.060653 & -0.024309 & & & -0.060653 & -0.024309 \\ & & & 0.016964 & & & 0.024309 & 0.007345 \\ & & & & 12 & 6 & & \\ & & & & & 4 & & \\ & & & & & & 0.060653 & 0.024309 \\ \text{symm.} & & & & & & & 0.016964 \end{bmatrix}$$

$$[g_{AB}] = \frac{P}{l}\begin{bmatrix} 1.2 & -0.1 & 0.09610 & -0.00787 & -1.2 & -0.1 & -0.09610 & -0.00787 \\ & 0.13333 & -0.00787 & 0.01032 & 0.1 & -0.03333 & 0.00787 & -0.00244 \\ & & 0.85926 & -0.06942 & -0.09610 & -0.00787 & -0.85926 & -0.06942 \\ & & & 0.08925 & 0.00787 & -0.00244 & 0.06942 & -0.01982 \\ & & & & 1.2 & 0.1 & 0.09610 & 0.00787 \\ & & & & & 0.13333 & 0.00787 & 0.01032 \\ & & & & & & 0.85926 & 0.06942 \\ \text{symm.} & & & & & & & 0.08925 \end{bmatrix}$$

The boundary conditions are

$$Q_2 = Q_4 = 0 \quad \text{and} \quad q_1 = q_3 = 0.$$

For symmetric (odd) buckling modes one also has

$$Q_5 = Q_7 = 0 \quad \text{and} \quad q_6 = q_8 = 0.$$

The condition of a nonvanishing displacement vector q leads

$$|\lambda\delta_{ij}-[g_{ij}]^{-1}[k_{ij}]| = 0$$

where δ_{ij} is the Kronecker symbol and

$$[\tilde{k}_{ij}] = \begin{bmatrix} 4 & 0 & 6 & 0 \\ & 0{\cdot}016964 & 0 & 0{\cdot}024309 \\ & & 12 & 0 \\ \text{symm.} & & & 0{\cdot}060653 \end{bmatrix}$$

$$[\tilde{g}_{ij}] = \begin{bmatrix} 0{\cdot}13333 & 0{\cdot}01032 & 0{\cdot}1 & 0{\cdot}00787 \\ & 0{\cdot}08925 & 0{\cdot}00787 & 0{\cdot}06942 \\ & & 1{\cdot}2 & 0{\cdot}09610 \\ \text{symm.} & & & 0{\cdot}85926 \end{bmatrix}$$

and

$$\lambda = Pl^2/EI_{xx}.$$

Due to the fact that the stiffness matrix K is uncoupled, it is easier to evaluate its inverse and consequently the eigenvalues Λ of the matrix $[k_{ij}]^{-1}[g_{ij}]$. It is obvious that the eigenvalues λ and Λ are related through $\Lambda^{-1} = \lambda$.

For the treated case, the two eigenvalues λ corresponding to the lowest bending and torsional buckling mode yield

$$P_{cr} = 4(2{\cdot}508)EI_{xx}/L^2 = 10{\cdot}03\ EI_{xx}/L^2$$

and

$$P_{cr} = 4(0{\cdot}0303)\ EI_{xx}/L^2 = 0{\cdot}121\ EI_{xx}/L^2.$$

The exact values as calculated from Ref. [1] are

$$P_{cr} = 9{\cdot}92\ EI_{xx}/L^2 \quad \text{and} \quad P_{cr} = 0{\cdot}119\ EI_{xx}/L^2.$$

Thus, they apparently compare favorably even when only two elements are used.

The critical force for the buckling in the plane of largest rigidity is

$$P_{cr} = \pi^2\ EI_{yy}/L^2.$$

9.3 *Free vibrations of a simply supported beam*

As the last example, the free vibrations of a simply supported beam presented in Fig. 7 will be examined. In addition to the already established matrix $[k]$, one should evaluate the submatrices $[m^{x\varphi}]$ and $[m^{\varphi\varphi}]$ using equations (6.12) and (6.73).

$$[m_{x\varphi}] = 2\rho A a_y \begin{bmatrix} 0{\cdot}0370836 & -0{\cdot}050096 & 0{\cdot}129165 & 0{\cdot}029264 \\ -0{\cdot}052406 & 0{\cdot}009003 & -0{\cdot}031093 & -0{\cdot}006763 \\ 0{\cdot}129165 & -0{\cdot}029264 & 0{\cdot}370836 & 0{\cdot}050096 \\ 0{\cdot}031093 & -0{\cdot}006763 & 0{\cdot}052406 & 0{\cdot}009003 \end{bmatrix}$$

$$[m_{\varphi\varphi}] = \rho\frac{Ar^2}{l} \begin{bmatrix} 0{\cdot}370861 & -0{\cdot}050429 & 0{\cdot}129137 & 0{\cdot}029674 \\ & 0{\cdot}008259 & -0{\cdot}028939 & -0{\cdot}006339 \\ & & 0{\cdot}370861 & 0{\cdot}049691 \\ \text{symm.} & & & 0{\cdot}008259 \end{bmatrix}$$

The total matrix $[m]$ for the element AB is thus,

$$[m] = \frac{\rho A l}{10}\begin{bmatrix} \underline{3{\cdot}71429} & -0{\cdot}52381 & 0{\cdot}29804 & -0{\cdot}04026 & 1{\cdot}28571 & 0{\cdot}30952 & 0{\cdot}10381 & 0{\cdot}02352 \\ & \underline{0{\cdot}09524} & -0{\cdot}04212 & 0{\cdot}00724 & -0{\cdot}30952 & -0{\cdot}07143 & -0{\cdot}02499 & -0{\cdot}00544 \\ & & \underline{2{\cdot}67539} & -0{\cdot}36379 & 0{\cdot}10381 & 0{\cdot}02499 & 0{\cdot}93159 & 0{\cdot}21407 \\ & & & \underline{0{\cdot}05958} & -0{\cdot}02352 & -0{\cdot}00544 & -0{\cdot}20877 & -0{\cdot}04573 \\ & & & & \underline{3{\cdot}71429} & 0{\cdot}52381 & 0{\cdot}29804 & 0{\cdot}04026 \\ & & & & & \underline{0{\cdot}09524} & 0{\cdot}04212 & 0{\cdot}00724 \\ & & & & & & \underline{2{\cdot}67539} & 0{\cdot}35847 \\ \text{symm.} & & & & & & & 0{\cdot}05958 \end{bmatrix}$$

For the same boundary conditions and for symmetric (odd) vibration modes one has

$$|\lambda\delta_{ij} - [\tilde{m}]^{-1}[\tilde{k}]| = 0$$

where

$$[\tilde{m}] = \begin{bmatrix} 0{\cdot}009524 & 0{\cdot}000724 & -0{\cdot}030952 & -0{\cdot}002499 \\ & 0{\cdot}005958 & -0{\cdot}002352 & -0{\cdot}020877 \\ & & 0{\cdot}371429 & 0{\cdot}029804 \\ \text{symm.} & & & 0{\cdot}267537 \end{bmatrix}$$

and

$$\lambda = \omega^2 m l^4 / EI_{xx}.$$

The two eigenvalues λ of $[m]_0^{-1}[k]$ corresponding to the two lowest torsional and bending modes are

$$\lambda = 6{\cdot}200 \quad \text{and} \quad \lambda = 765{\cdot}696$$

Thus, the two natural frequencies corresponding to lowest bending and torsional modes are

$$\omega_x = 9{\cdot}960\,(EI_{xx}/mL^{y})^{\frac{1}{2}} \qquad \omega_\varphi = 3062{\cdot}784\,(EI_{xx}/mL^4)^{\frac{1}{2}}$$

The vibrations in the vertical direction with the lowest natural frequency

$$\omega_y = \pi^2\,(EI_{yy}/mL^4)^{\frac{1}{2}}$$

are again uncoupled.

10. CONCLUSIONS AND ANALYSIS OF THE RESULTS

The presented matrix formulation for the static and dynamic analysis of the structures assembled from thin-walled members is by itself rather general and consistent. In the same time, it is only an extension of already established procedures for structures assembled from solid members.

A number of related linearized problems such as lateral buckling, second order theory, parametric resonance, etc. may be also treated once the stiffness, stability and mass matrix are established. The discussion of these problems appears to be redundant in wake of numerous works concerned with same phenomena in case of solid beam structures.

There is, however, one additional thing we would like to discuss. In case of linear static problems, the solution presented herein is exact, since the displacement modes γ_i are the solutions of the governing homogeneous equation (2.5d). For the dynamic and stability problems, the story is somewhat different. The choice of displacement modes is a matter of our assumptions and thus, subject to discussion. The presented paper is based on the assumption that the static deformation modes are similar in shape to buckling and dynamic modes. This assumption was frequently successfully employed, in conjunction with variational methods, long before the advent of the finite elements method. This would, however, be a somewhat less convincing assertion if not supported by few numbers.

In order to analyse the merits of proposed technique let us present a brief stability analysis of a simply supported thinwalled member with two axes of symmetry

$$(a_x = a_y = \beta_x = \beta_y = 0)$$

subjected to an axial load $P(M_x = M_y = 0)$. For this, simplest of all cases, one finds the exact value of the first buckling load [1] to be:

$$P_\Omega = (\pi^2 + \kappa^2)EI_{\Omega\Omega}/r^2l^2. \tag{9.1}$$

It is fairly obvious that the choice of the cubic polynomials satisfying boundary conditions as deformation modes γ_i essentially prescribes $\kappa = 0$ (see Section 7). No matter how many elements we use, our result will not be able to reflect the second term in parentheses in equation (9.1). It will always give results close to

$$P_\Omega = \pi^2 EI_{\Omega\Omega}/r^2l^2 \tag{9.2}$$

which may be well under the exact value for every κ not being close to zero.

If we use the admittedly more complicated functions $\gamma_i(z)$ presented by this paper, results are extremely close even with only two elements and for any κ which makes sense. Table 1 demonstrates the achieved accuracy.

TABLE 1. BUCKLING FORCE FOR A SIMPLY SUPPORTED MEMBER WITH TWO AXES OF SYMMETRY

P_Ω / kl	Exact solution	Finite element solution: Proposed $\gamma_i(f)$	Finite element solution: Cubic polynomial for $\gamma_i(f)$	Multiplicator
0	9·87	9·92 (0·5)†	9·92	
1	13·87	14·01 (1·0)	9·92	
2	25·87	26·30 (1·6)	9·92	$EI_{\Omega\Omega}/r^2L^2$
3	45·87	46·99 (2·4)	9·92	
4	73·87	76·13 (3·0)	9·92	
5	109·87	113·65 (3·3)	9·92	
10	409·87	423·20 (3·1)	9·92	

† In column 3, the percentage difference is given in parentheses. Note that the method yields always an upper bound to the exact solution which one expects as a result of employed variational principle.

In conclusion, it is our opinion that simpler deformation modes than ones proposed herein may be constructed for specific narrow ranges of parameter κ. There is, however, no evidence that a simple set of, say, cubic polynomials may yield results of accuracy comparable with ones obtained herein.

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Абстракт—Предлагается последоватеьная матричная формулировка расчета дискретных злементов для линейных задач и для собственных значений, которые применяются в конструкциях, состоящих с тонкостенных сегментов откритого профиля. Используя решения однородного дифференциальн-ого уравнения, касающегося статической задачи, выводятся формы деформации и матрица козф-фициентов жесткости, нагрузки, устойчивости и массы. Процесс иллюстрируется соответствующими примерами, касающимися изгиба одновременно со сдвигом, устойчивости и вибрации. Полученные результаты точны для случая линейных статических задач и очень близки верхнего предела, в задачах для собственных значений.

Вследствие оганичения процесса показано, что предлагаемый подход является развитием метода, используемого для сплошных стержневых конструкций.

APPENDIX

The load matrix

$$
\begin{bmatrix} Q_9 \\ Q_{10} \\ Q_{11} \\ Q_{12} \end{bmatrix}
=
\begin{bmatrix}
\frac{1}{\kappa^2}\left(-1+\frac{\kappa^2}{4}+\frac{\kappa^3}{12}\frac{\mathrm{sh}\,\kappa}{D}\right) & \frac{1}{\kappa^2}\left(1+\frac{\kappa^2}{4}+\frac{\kappa^3}{12}\frac{\mathrm{sh}\,\kappa}{D}\right) & 1/2l & 1/2l & \frac{\gamma_9(x_0)}{l} & \frac{-\kappa\gamma_9'(x_0)}{l^2} \\
\begin{array}{l}\frac{1}{D\kappa^2}\Big[2(1-\mathrm{ch}\,\kappa)+\frac{3\kappa}{2}\mathrm{sh}\,\kappa \\ \quad -\frac{\kappa^2}{6}(1+2\,\mathrm{ch}\,\kappa)\Big]\end{array} & \frac{1}{D\kappa^2}\left(\frac{\kappa}{2}\mathrm{sh}\,\kappa-\frac{\kappa^2}{3}-\frac{\kappa^2}{6}\mathrm{ch}\,\kappa\right) & \begin{array}{l}\frac{-1}{D\kappa^2 l}\Big[2(1-\mathrm{ch}\,\kappa)+2\kappa\,\mathrm{sh}\,\kappa \\ \quad -\frac{\kappa^2}{2}(1+\mathrm{ch}\,\kappa)\Big]\end{array} & \begin{array}{l}\frac{-1}{D\kappa^2 l}\Big[2(1-\mathrm{ch}\,\kappa)+2\kappa\,\mathrm{sh}\,\kappa \\ \quad -\frac{\kappa^2}{2}(1+\mathrm{ch}\,\kappa)\Big]\end{array} & \frac{\gamma_{10}(x_0)}{l} & \frac{\kappa\gamma_{10}'(x_0)}{-l^2} \\
\frac{1}{\kappa^2}\left(1+\frac{\kappa^2}{4}+\frac{\kappa^3}{12}\frac{\mathrm{sh}\,\kappa}{D}\right) & \frac{1}{\kappa^2}\left(-1+\frac{\kappa^2}{4}+\frac{\kappa^3}{12}\frac{\mathrm{sh}\,\kappa}{D}\right) & -1/2l & -1/2l & \frac{\gamma_{11}(x_0)}{l} & \frac{\kappa\gamma_{11}'(x_0)}{-l^2} \\
\frac{1-}{D\kappa^2}\left(\tfrac{1}{2}\kappa\,\mathrm{sh}\,\kappa-\frac{\kappa^2}{3}-\frac{\kappa^2}{6}\mathrm{ch}\,\kappa\right) & \begin{array}{l}\frac{-1}{D\kappa^2}\Big[2(1-\mathrm{ch}\,\kappa)+\frac{3\kappa}{2}\mathrm{sh}\,\kappa \\ \quad -\frac{\kappa^2}{6}(1+2\,\mathrm{ch}\,\kappa)\Big]\end{array} & \begin{array}{l}\frac{1}{D\kappa^2 l}\Big[2(1-\mathrm{ch}\,\kappa)+2\kappa\,\mathrm{sh}\,\kappa \\ \quad -\frac{\kappa^2}{2}(1+\mathrm{ch}\,\kappa)\Big]\end{array} & \begin{array}{l}\frac{1}{D\kappa^2 l}\Big[2(1-\mathrm{ch}\,\kappa)+2\kappa\,\mathrm{sh}\,\kappa \\ \quad -\frac{\kappa^2}{2}(1+\mathrm{ch}\,\kappa)\Big]\end{array} & \frac{\gamma_{12}(x_0)}{l} & \frac{\kappa\gamma_{12}'(x_0)}{-l^2}
\end{bmatrix}
\begin{bmatrix} m_D^L \\ m_D^R \\ m_\Omega^L \\ m_\Omega^R \\ T^* \\ M_\Omega^* \end{bmatrix}
$$